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# Gauge-Invariant Chiral Schwinger Model With Faddeevian Regularization: Stueckelberg Term, Operator Solution, and Hamiltonian and BRST Formulations

**Usha Kulshreshtha[1,2]**

A chiral Schwinger model with the Faddeevian regularization à la Mitra is studied in one-space one-time dimension in the conventional form of dynamics (on the hyperplanes $x^0 =$ constant) called the "Instant-Form" (IF) dynamics. The original IF theory is seen to be gauge-noninvariant (GNI). Corresponding to this GNI model, a gauge-invariant (GI) theory is constructed through the so-called Stueckelberg term. The operator solution and the Hamiltonian and BRST formulations of the resulting GI theory, obtained by the inclusion of the Stueckelberg term in the action of the original GNI theory, are then investigated with some specific gauge choices. The physical contents of the original GNI theory are also recovered from the newly constructed GI theory under a special gauge.

## 1. INTRODUCTION

The Schwinger model describing electrodynamics in one-space one-time ((1 + 1)-) dimension with massless fermions and its chiral versions called the chiral Schwinger models (CSMs) have been of a very wide interest in recent years (Boyanovski, 1987; Falck and Kramer, 1987; Floreanini and Jackiw, 1987; Girotti *et al.*, 1986; Harada, 1990a,b; Harada and Tsutsui, 1988; Jackiw and Rajaraman, 1985; Kim *et al.*, 1990, 1991, 1992; Kulshreshtha, 2000; Kulshreshtha *et al.*, 1993a,b,c, 1994a,b,c, 1999; Kulshreshtha and Kulshreshtha, 1998, in press; Kulshreshtha and Mueller-Kirsten, 1992; Mitra, 1992; Mitra and Rajaraman, 1988; Mukhopadhyay and Mitra, 1995, 1995a,b; Rajaraman, 1985; Schwinger, 1962). The CSMs describe a massless Dirac field $\psi(x, t)$ in (1 + 1)-dimension with only one of its chiral components coupled to a $U(1)$ vector gauge field $A^{\mu}(x, t)$ (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962).

[1] Fachbereich Physik der Universität Kaiserslautern, Germany.
[2] Department of Physics, University of Delhi, Delhi-110007, India; e-mail: usha@physik.uni-kl.de, usha@physics.du.ac.in.



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One of the remarkable achievements of the studies of such theories has been the development of the fermion–boson correspondence in two-dimensional quantum field theories. The other important achievement has been in the field of understanding the phenomena of gauge-anomalies and the gauge-anomalous field theories (Boyanovski, 1987; Falck and Kramer, 1987; Floreanini and Jackiw, 1987; Girotti *et al.*, 1986; Harada, 1990a,b; Harada and Isutsui, 1988; Jackiw and Rajaraman, 1985; Kim *et al.*, 1990, 1991, 1992; Kulshreshtha, 2000; Kulshreshtha *et al.*, 1993a,b,c, 1994a,b,c, 1999; Kulshreshtha and Kulshreshtha, 2001, in press; Kulshreshtha and Mueller-Kirsten, 1992; Mitra, 1992; Mitra and Rajaraman, 1988; Mukhopadhyay and Mitra, 1995, 1995a,b; Rajaraman, 1985; Schwinger, 1962).

Jackiw and Rajaraman (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962), in particular, have considered a gauge-anomalous CSM (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962). By studying the field equations and propagator obtained from the effective gauge field action, they concluded that the theory was not gauge-invariant (GI) but was unitary and amenable to particle interpretation (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962). They also found that the vector gauge boson necessarily acquires a mass when consistency and unitarity are demanded (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962). The class of regularizations that have been considered involve the dimensionless Jackiw–Rajaraman (JR) regularization parameter a (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962); the JR-CSM (Boyanovski, 1987; Falck and Kramer, 1987; Floreanini and Jackiw, 1987; Girotti *et al.*, 1986; Harada, 1990a,b; Harada and Isutsui, 1988; Jackiw and Rajaraman, 1985; Kim *et al.*, 1990, 1991, 1992; Kulshreshtha *et al.*, 1993a,b,c, 1994a,b,c, 1999; Kulshreshtha and Mueller-Kirsten, 1992; Mitra and Rajaraman, 1988; Rajaraman, 1985; Schwinger, 1962) is found to admit exact solutions in a positive metric Hilbert space respecting unitarity for $a \geq 1$, for which the theory is sensible (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962). In fact, the model is seen to yield a sensible theory for a class of regularizations (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962). The spectrum of the theory depends on the regularization in a crucial way and it is seen to contain, for $a > 1$, a massive photon in addition to a massless fermion, and for $a = 1$, only a massless fermion (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962).

A new regularization that does not belong to the above class has recently been considered by Mitra (Mitra, 1992). With this regularization (Mitra, 1992), the photon is once again massive and the massless fermion present in the theory has (in contrast to the JR regularization) a chirality opposite to that entering the interaction with the electromagnetic field (Mitra, 1992). This regularization has been called by Mitra (Mitra, 1992) as the Faddeevian regularization (Mitra, 1992;

Mukhopadhyay and Mitra, 1995a,b). The original theory (à la Mitra, Mitra, 1992) (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b), which is in accordance with the Faddeev's picture (Faddeev, 1984; Faddeev and Shatashvili, 1986) of gauge-anomalous theories, has a masslike term for the vector gauge boson $A_\mu$ different from those of the class of models (called JR-CMS) studied earlier (Boyanovski, 1987; Falck and Kramer, 1987; Girotti *et al.*, 1986; Jackiw and Rajaraman, 1985; Kulshreshtha *et al.*, 1993a, 1994a,b; Mitra and Rajaraman, 1988; Rajaraman, 1985; Schwinger, 1962) and may be taken as a signature of new regularization (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b).

This theory in contrast with the JR-CSM is seen to possess a self-dual boson that could also be thought of as a chiral fermion. The theory possesses a vector gauge anomaly and lacks the gauge invariance. This, in fact, is a consequence of the Faddeev's anomaly (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b) in the theory (where the matrix of the poisson brackets of the constraints of the theory becomes nonsingular because of the nonvanishing poisson bracket of the Gauss law constraint of the theory with itself, called the Faddeev's anomaly (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b)), which leads to the breaking of the vector gauge symmetry of the theory, making it gauge-anomalous. The Hamiltonian formulation (Dirac, 1950) of this theory has been studied in the "Instant Form" (IF) (Dirac, 1949) by Mitra ((Mitra, 1992), where the theory is seen to be gauge-noninvariant (GNI), possessing a set of three second-class constraints (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b).

In a recent paper (Kulshreshtha, 1998), we have studied this theory on the hyperplanes of the light front ($x^0 + x^1 =$ constant (Dirac, 1949)) describing the "Front Form" (FF) of dynamics (Kulshreshtha, 1998). In the present work we study this theory in the conventional form of dynamics on the hyperplanes ($x^0 =$ constant (Dirac, 1949)), called the instant-form theory, derived from the original GNI theory (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b) through the so-called Stueckelberg term (Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a, 1994a,b,c; Stueckelberg, 1941, 1957): the addition of which to the action of the GNI theory restores the gauge symmetry of the theory. The Hamiltonian and Becchi, Rouet, Stora, and Tyutin (BRST) formulations (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998, in press; Kulshreshtha *et al.*, 1993a,b,c, 1994a,b,c, 1999; Kulshreshtha and Kulshreshtha, 2001, in press; Kulshreshtha and Mueller-Kirsten, 1992; Nameshchansky *et al.*, 1988; Tyutin, 1975) and the operator solutions of this newly constructed GI theory are then studied under some specific gauge choices. The physical contents of the original GNI theory are also recovered under a special choice of gauge (Falck and Kramer, 1987), and the equivalence of the quantized GI and GNI theories is established (Falck and Kramer, 1987).

In the usual Hamiltonian formulation of a GI theory under some gauge-fixing conditions, one necessarily destroys the gauge invariance of the theory by fixing the gauge (which converts a set of first-class constraints into a set of second-order constraints, implying a breaking of gauge invariance under the gauge fixing.) To achieve the quantization of a GI theory such that the gauge invariance of the theory is maintained even under gauge fixing, one goes to a more generalized procedure called the BRST formulation (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 2000; Kulshreshtha *et al.*, 1993a,b,c, 1994a,b,c, 1999; Kulshreshtha and Kulshreshtha, 2001, in press; Kulshreshtha and Mueller-Kirsten, 1992; Nameshchansky *et al.*, 1988; Tyutin, 1975). In the BRST formulation of a GI theory, the theory is rewritten as a quantum system that possesses a generalized gauge invariance called the BRST symmetry. For this, one enlarges the Hilbert space of the GI theory and replaces the notion of the gauge transformation, which shifts operators by $c$-number of the gauge functions, namely by a BRST transformation, which mixes operators having different statistics. In view of this, one introduces new anticommuting variables $c$ and $\bar{c}$ called the Faddeev–Popov ghost and antighost fields, which are Grassmann numbers on the classical level and operators in the quantized theory, and a commuting variable $b$ called the Nakanishi–Lautrup field (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 2000; Kulshreshtha *et al.*, 1993a,b,c, 1994a,b,c, 1999; Kulshreshtha and Kulshreshtha, 2001; Kulshreshtha and Mueller-Kirsten, 1992; Nameshchansky *et al.*, 1988; Tyutin, 1975). In the BRST formulation, one thus embeds a GI theory into a BRST-invariant system, and quantum Hamiltonian of the system (which includes the gauge-fixing contribution) commutes with the BRST charge operator $Q$ as well as with the anti-BRST charge operator $\bar{Q}$, the new symmetry of the quantum system (the BRST symmetry) that replaces the gauge invariance is maintained (even under gauge fixing) and hence projecting any state onto the sector of BRST- and anti-BRST-invariant states yields a theory that is isomorphic to the original GI theory. The unitarity and consistency of the BRST-invariant theory described by the gauge-fixed quantum Lagrangian is guaranteed by the conservation and nilpotency of the BRST charge $Q$.

The plan of the paper is as follows: In Section 2, we briefly recapitulate the CSM with the Faddeevian regularization (Kulshreshtha, 1998; Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b). In Section 3, we construct the GI theory corresponding to the original GNI theory (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b) through the Stueckelberg term (Stueckelberg, 1941, 1957). The Hamiltonian formulation and the operator solutions of this newly constructed GI theory (obtained by the inclusion of the Stueckelberg term in the action of the original GNI theory) are studied in Section 4, where the contents of the original GNI theory are also recovered from that of the GI theory under a special choice of gauge (Falck and Kramer 1987). Finally, the BRST formulation of the GI theory is studied in Section 5, and the conclusions and discussions are given in Section 6.

## 2. THE GNI THEORY À LA MITRA (MITRA, 1992; MUKHOPADHYAY AND MITRA, 1995A,B)

In this section, we briefly recapitulate the basics of the GNI-CSM with the Faddeevian regularization due to Mitra (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b) in one-space one-time dimension. In Kulshreshtha (1998), this theory has been studied on the hyperplanes of the "Light-Front" ($x^0 + x^1 = \text{constant}$), describing the "Front Form" of dynamics (Dirac, 1949). In the present work we study this theory in the conventional form of dynamics on the hyperplanes ($x^0 = \text{constant}$), called the "Instant Form" of dynamics (Dirac, 1949). The theory in the instant form (Dirac, 1949) is described by the bosonized action (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b):

$$S^{\mathrm{N}} = \int \mathscr{L}^{\mathrm{N}} dx_0\, dx_1 \tag{2.1a}$$

$$\begin{aligned}\mathscr{L}^{\mathrm{N}} &= \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + e(g^{\mu\nu} - \varepsilon^{\mu\nu})(\partial_\mu\phi)A_\nu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right.\\ &\quad \left. + \left\{\frac{1}{2}e^2 A_\mu M^{\mu\nu} A_\nu\right\}\right] \end{aligned}\tag{2.1b}$$

$$\begin{aligned}&= \left[\frac{1}{2}(\mathring{\phi}^2 - \phi'^2) + e(\mathring{\phi} + \phi')(A_0 - A_1) + \frac{1}{2}(\mathring{A}_1 - A_0')^2\right.\\ &\quad \left. + \left\{\frac{1}{2}e^2(A_0 - A_1)^2 - 2e^2 A_1^2\right\}\right]\end{aligned}\tag{2.1c}$$

$$g^{\mu\nu} = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} \quad M^{\mu\nu} = \begin{bmatrix} 1 & -1 \\ -1 & -3 \end{bmatrix} \quad \varepsilon^{\mu\nu} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \tag{2.1d}$$

The overdots and primes denote time and space derivatives respectively. The first term in (2.1) represents (Mitra, 1992) a massless boson that is equivalent to a massless fermion in two dimensions. The second term represents the chiral coupling of this fermion to the electromagnetic field $A_\mu$. The third term is the kinetic energy term of the electromagentic field. The last term in (2.1), namely,

$$\mathscr{L}_{\mathrm{m}} = \frac{1}{2}e^2 A_\mu\, M^{\mu\nu} A_\nu \tag{2.2a}$$

$$= \frac{1}{2}e^2(A_0 - A_1)^2 - 2e^2\, A_1^2 \tag{2.2b}$$

is the mass term for the electromagnetic field $A_\mu$ and has been derived explicitly in Mukhopadhyay and Mitra (1995a), using the Pauli–Villars method of regularization, where the effective action $S^{\mathrm{N}}$ has been obtained with this unconventional mass-term (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b). It corresponds to a new class of regularization called the Faddeevian regularization (Faddeev, 1984; Faddeev and Shatashvili, 1986); this theory is in accordance with the Faddeev's picture of gauge-anomalous theories (Faddeev, 1984; Faddeev and Shatashvili, 1986; Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b). It is important to recall that in a true or bonafide GI theory the matrix of the poisson brackets (PBs) of the constraints is a null matrix. Faddeev (Faddeev, 1984; Faddeev and Shatashvili, 1986) visualised a situation where anomalies make the PB of the Gauss law constraint with itself nonvanishing. If this happens, the set of constraints becomes second-class and the gauge invariance of the theory gets lost (Mitra, 1992; Faddeev, 1984; Faddeev and Shatashvili, 1986). Faddeev argued that there would be more physical degrees of freedom in such a theory than in the case of GI theories because no gauge-fixing conditions would be needed to quantize the theory. In the present CSM with the new (so-called Faddeevian) regularization considered in previous works (Faddeev, 1984; Faddeev and Shatashvili, 1986; Kulshreshtha, 1998; Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b), the Faddeev's mechanism works: namely, the constraints become second-class through an anomaly in the PB of the Gauss law constraint with itself (Mitra, 1992). It may be worthwhile to record the mass term for the vector gauge field $A_\mu$, for the JR-CSM (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962)

$$\widetilde{\mathscr{L}}_{\rm m} = \frac{1}{2} a\, e^2 A_\mu\, A^\mu \tag{2.3}$$

It corresponds to the so-called JR or standard regularization (Jackiw and Rajaraman, 1985; Rajaraman, 1985; Schwinger, 1962), where $a$ is the regularization parameter introduced by JR. The mass term for $A_\mu$, in fact, arises from the regularization ambiguities associated with the definition of current and contains the fermionic one-loop effects. It is thus obvious that the present theory (with the Faddeevian regularization) (Faddeev, 1984; Faddeev and Shatashvili, 1986; Kulshreshtha, 1998; Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b) corresponds to a regularization different from those involved in the class of JR-CSMs (Boyanovski, 1987; Falck and Kramer, 1987; Girotti *et al.*, 1986; Jackiw and Rajaraman, 1985; Kulshreshtha *et al.*, 1993a, 1994a,b; Mitra and Rajaraman, 1988; Rajaraman, 1985; Schwinger, 1962) and the mass term $\mathscr{L}_{\rm m}$ (2.2) is regarded here as a signature of the new regularization (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b).

Further, the mass-like term for the vector gauge field $A_\mu$ in $\mathscr{L}^{\rm N}$ (namely $\mathscr{L}_{\rm m}$ (2.2)) does not have the Lorentz invariance and therefore the theory lacks manifest Lorentz covariance. However, the Poincaré generators of the theory all defined on the constraints hypersurface are seen to satisfy the Poincaré algebra (Mitra, 1992):

$$\left[P^0_{\rm R},\, p^1_{\rm R}\right] = 0 \tag{2.4a}$$

$$\left[M_{\mathrm{R}}^{10}, P_{\mathrm{R}}^{0}\right] = -i P_{\mathrm{R}}^{1} \tag{2.4b}$$

$$\left[M_{\mathrm{R}}^{10}, P_{\mathrm{R}}^{1}\right] = -i P_{\mathrm{R}}^{0} \tag{2.4c}$$

where $P_{\mathrm{R}}^{0}(\equiv H_{\mathrm{R}})$, $P_{\mathrm{R}}^{1}(\equiv P_{\mathrm{R}})$ and $M_{\mathrm{R}}^{10}(\equiv M_{\mathrm{R}})$ are the field operators corresponding to the field energy or the Hamiltonian, field momentum, and the Lorentz-boost generator of the theory (defined by $S^{\mathrm{N}}$ or $\mathscr{L}^{\mathrm{N}}$ (2.1)) defined on the constraints hypersurface (Mitra, 1992). In view of this, the theory defined by $\mathscr{L}^{\mathrm{N}}$, despite the lack of manifest Lorentz covariance, is seen to be implicitly Lorentz-invariant (Mitra, 1992; Kulshreshtha, 1998).

Using the Euler–Lagrange field equations of motion obtained from $\mathscr{L}^{\mathrm{N}}$ (2.1), it is easy to see that the divergence of the vector gauge current for the theory defined by $\mathscr{L}^{\mathrm{N}}$ does not vanish, implying that the theory possesses a vector gauge anomaly or that the theory is GNI.

A study of the canonical structure of the theory reveals that it possesses a set of three second-class constraints (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b):

$$\Omega_1 = \Pi_0 \approx 0 \tag{2.5a}$$

$$\Omega_2 = (\partial_1 E + e\,\Pi + e\,\partial_1\phi) \approx 0 \tag{2.5b}$$

$$\Omega_3 = (A_0 + A_1) \approx 0 \tag{2.5c}$$

Where $\Omega_1$ is a primary constraint and $\Omega_2$ and $\Omega_3$ are secondary constraints. Here $\Pi$, $\Pi_0$, and $E$ $(\equiv\Pi^1)$ are the momenta cononically conjugate respectively to $\phi$, $A_0$, and $A_1$. The matrix of the PBs of the constraints $\Omega_i$ is seen to be nonsingular, implying that the set of constraints $\Omega_i$ is second-class, reflecting a lack of gauge-invariance in the theory and consequently the model describes a GNI or gauge-anomalous theory (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b). The gauge symmetry, however, when present in a theory has many beneficial consequences. It is rather well known that the addition of an appropriate Stueckelberg-/ Wess–Zumino-kind of a term to the action of a GNI theory possessing a set of second-class constraints converts it into a GI theory possessing a set of first-class constraints. Under some special choice of gauge, it is, however, possible to recover the physical content of the GNI theory from the GI theory (Boyanovski, 1987; Girotti *et al.*, 1986; Mitra and Rajaraman, 1988). In the next section, we would construct a GI theory corresponding to the present GNI theory (described by $\mathscr{L}^{\mathrm{N}}$ (2.1)) à la Mitra (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b).

Further, using the Hamilton's equations of motion of the GNI theory that preserve the constraints of the theory (2.5) in the course of time, one can see that $A_1$ satisfies the Klein–Gordon equation (Mitra, 1992)

$$(\partial_\mu \partial^\mu + 4\,e^2)A_1 = 0 \tag{2.6}$$

implying that the photon has a mass equal to $2|e|$. Also, by defining a new field $\chi$ as (Mitra, 1992; Mukhopadhyay and Mitra, 1995,a,b)

$$\chi = \phi + \frac{1}{2e}(\partial_0 A_1 + \partial_1 A_1) \tag{2.7}$$

It is seen that the field $\chi$ satisfies the antiduality condition (Kulshreshtha, 1998; Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b):

$$\partial_0 \chi + \partial_1 \chi = 0 \tag{2.8}$$

implying that $\chi$ is a self-dual field, and thereby implying that the theory contains a chiral boson, which could be thought of as a chiral fermion (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b). The fields $\phi$, $A_0$, and $A_1$ could then be expressed in terms of the free massive scalar field $A_1$ and the free self-dual boson $\chi$ (or equivalently a chiral fermion) (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b). Thus the spectrum of the theory in this Faddeevian regularization is found to contain a self-dual boson (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b). This is in contrast to the CSM with the JR regularization (i.e. the JR-CSM) (Boyanovski, 1987; Falck and Kramer, 1987; Girotti *et al.*, 1986; Jackiw and Rajaraman, 1985; Kulshreshtha *et al.*, 1993a, 1994a,b; Mitra and Rajaraman, 1988; Rajaraman, 1985; Schwinger, 1962) Also, for later purposes, it may be worthwhile to record the nonvanishing equal-time commutators of this GNI theory obtained by the Dirac quantization of the theory:

$$[\phi(x), \Pi(y)] = \frac{3}{2} i \delta(x-y) \tag{2.8a}$$

$$[A_1(x), \Pi(y)] = \frac{-i}{2e} \delta(x-y) \tag{2.8b}$$

$$[A_0(x), \Pi(y)] = \frac{-i}{2e} \partial_1 \delta(x-y) \tag{2.8c}$$

$$[A_1(x), E(y)] = i\delta(x-y) \tag{2.8d}$$

$$[A_0(x), E(y)] = -i\delta(x-y) \tag{2.8e}$$

$$[\phi(x), \phi(y)] = \frac{+i}{4} \epsilon(x-y) \tag{2.8f}$$

$$[\phi(x), A_1(y)] = \frac{-i}{2e} \delta(x-y) \tag{2.8g}$$

$$[\phi(x), A_0(y)] = \frac{-i}{2e} \delta(x-y) \tag{2.8h}$$

$$[A_1(x), A_1(y)] = \frac{i}{2e^2} \partial_1 \delta(x-y) \tag{2.8i}$$

$$[A_1(x), A_0(y)] = \frac{i}{2e^2}\partial_1\delta(x-y) \tag{2.8j}$$

$$[A_0(x), A_0(y)] = \frac{i}{2e^2}\partial_1\delta(x-y) \tag{2.8k}$$

$$[\Pi(x), \Pi(y)] = \frac{i}{2}\partial_1\delta(x-y) \tag{2.8l}$$

Here $\epsilon(x-y)$ is a step function defined as

$$\epsilon(x-y) = \begin{cases} +1, & (x-y) > 0 \\ -1, & (x-y) < 0 \end{cases} \tag{2.9}$$

### 2.1. The Theory in the Light-Front Frame

In a recent paper (Kulshreshtha, 1998) we have studied this theory in the light-front (LF) frame, on the hyperplanes of the LF: $\sqrt{2}x^+ = (x^0 + x^1) =$ constant. The main results of this work are briefly recapitulated here in this section. The theory, described in the LF frame by the Lagrangian density (2.16), with $\mu, \nu = +, -$, is seen to possess a set of three second-class constraints:

$$\rho_1 = \Pi^+ \approx 0, \qquad \rho_2 = [\Pi - \partial_-\phi - 2eA^+] \approx 0 \tag{2.10a}$$

$$\rho_3 = [\partial_-\Pi^- - 2e^2(A^- - A^+)] \approx 0 \tag{2.10b}$$

where $\rho_1$ and $\rho_2$ are primary constraints and $\rho_3$ is a secondary Gauß law constraint and $\Pi, \Pi^+$, and $\Pi^-$ are the canonical momenta conjugate respectively to the fields $\phi$, $A^-$, and $A^+$. The matrix of the PBs of these constraints $\rho_i$ is nonsingular, implying that the set of $\rho_i$ is second-class and that the corresponding theory lacks gauge invariance. An appropriate Stueckelberg term (ST) for this theory has been calculated in Kulshreshtha (1998) and it reads as

$$\mathcal{L}^S = [(1-2e+2e^2)(\partial_+\theta)(\partial_-\theta) - (1-2e)(\partial_+\phi)(\partial_-\theta) - (\partial_-\theta)(\partial_-\phi) + 2e(e-1)A^+(\partial_+\theta) - e^2(\partial_+\theta)^2 - 2e^2A^-(\partial_+\theta - \partial_-\theta)] \tag{2.11}$$

where $\theta$ is the Stueckelberg scalar field. The new theory obtained by the addition of this ST (2.11) to the Lagrangian density (2.1b) of the GNI theory is seen to be GI possessing a set of three first-class constraints (Kulshreshtha, 1998):

$$\chi_1 = \Pi^+ \approx 0, \qquad \chi_2 = [\Pi - \partial_-\phi - 2eA^+ + (1-2e)\partial_-\theta] \approx 0 \tag{2.12a}$$

$$\chi_3 = [\partial_-\Pi^- + \Pi_\theta - (1-2e)(\partial_-\theta) + \partial_-\phi + 2eA^+] \approx 0 \tag{2.12b}$$

Also, $\chi_1$ and $\chi_2$ here are the primary constraints of the new GI theory, and $\chi_3$ is the Gauß law secondary constraint. The matrix of the PBs of the constraints $\chi_i$ is singular, implying that the set $\chi_i$ is first-class and that the theory is GI. The divergence of the vector gauge current of the theory also vanishes, showing that the theory has at the classical level a local vector gauge symmetry. The action of the theory is indeed seen to be invariant under the local vector gauge transformations (LVGT) (Kulshreshtha, 1998):

$$\delta A^{+} = \partial_{-}\beta, \quad \delta A^{-} = \partial_{+}\beta, \quad \delta\phi = -\beta, \quad \delta\theta = -\beta \tag{2.13a}$$

$$\delta\Pi^{+} = \delta\Pi^{-} = \delta\Pi = \delta\Pi_{\theta} = 0; \qquad \beta = \beta(x^{+}, x^{-}) \tag{2.13b}$$

This new GI could therefore be quantized under some suitable light-cone gauges. However, for recovering the physical content of the original GNI theory described by (2.1b), we go to a unitary gauge $\partial^{\mu}\theta = 0$ and choose the gauge-fixing conditions

$$\nu_1 = -\partial_{-}\theta = 0; \qquad \nu_2 = [\Pi_{\theta} + \partial_{-}\phi - 2e(e-1)A^{+} + 2e^2 A^{-}] = 0 \tag{2.14}$$

The gauge $\nu_i$ translates the new GI theory into the original GNI theory and the physical content of the new GI theory under this gauge $\nu_i$ is the same as that of the original GNI theory (2.1b). In Kulshreshtha (1998) this new GI theory has been Hamiltonian- and BRST-quantized under some specific gauge choices (cf. Kulshreshtha, 1998, for the details).

## 3. CONSTRUCTION OF THE GI THEORY: THE STUECKELBERG TERM

For constructing a GI theory corresponding to $S^{\mathrm{N}}$ (2.1), we calculate the Stueckelberg term for $S^{\mathrm{N}}$. For this, we enlarge the Hilbert space of the GNI theory defined by (2.1), and introduce a new field $\theta$, called the Stueckelberg field (Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a, 1994a,b,c; Stueckelberg, 1941, 1957), through the following redefinition of fields $\phi$ and $A^{\mu}$ in the original action $S^{\mathrm{N}}$ (the motivation for which comes from the gauge transformations (4.7) of the expected GI theory (3.2)) (Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a, 1994a,b,c; Stueckelberg, 1941, 1957):

$$\phi \rightarrow \Phi = \phi - \theta \quad \text{and} \quad A^{\mu} \rightarrow \mathscr{A}^{\mu} = A^{\mu} + \partial^{\mu}\theta \tag{3.1}$$

Here, the Stueckelberg field $\theta$ is a full quantum field. Performing the changes (3.1) in $S^{\mathrm{N}}$ (2.1), we obtain the modified action as

$$S^{\mathrm{I}} (= S^{\mathrm{N}} + S^{\mathrm{S}}) = \int \mathscr{L}^{\mathrm{I}} dx_0\, dx_1 = \int (\mathscr{L}^{\mathrm{N}} + \mathscr{L}^{\mathrm{S}})\, dx_0\, dx_1 \tag{3.2a}$$

$$\mathscr{L}^{\mathrm{I}} = \mathscr{L}^{\mathrm{N}} + \mathscr{L}^{\mathrm{S}} \tag{3.2b}$$

$$S^{\mathrm{S}} = \int \mathscr{L}^{\mathrm{S}} dx_0\, dx_1 \tag{3.2c}$$

$$\mathscr{L}^{\mathrm{S}} = \Bigg[\frac{1}{2}(1-2e)\{(\partial_0\theta)^2 - (\partial_1\theta)^2\} - \{(\partial_0\phi)(\partial_0\theta) - (\partial_1\phi)(\partial_1\theta)\}$$
$$+ e(\partial_0\phi + \partial_1\phi)(\partial_0\theta - \partial_1\theta) - e(A_0 - A_1)(\partial_0\theta + \partial_1\theta)$$
$$+ \frac{1}{2}e^2(\partial_0\theta - \partial_1\theta)^2 + e^2(A_0 - A_1)(\partial_0\theta - \partial_1\theta) - 2e^2(\partial_1\theta)^2$$
$$- 4e^2 A_1(\partial_1\theta)\Bigg] \tag{3.2d}$$

Here $\mathscr{L}^{\mathrm{N}}$ and $S^{\mathrm{N}}$ are defined by (2.1) and $S^{\mathrm{S}}$ is the appropriate Stueckelberg term corresponding to the GNI action $S^{\mathrm{N}}$ (2.1). We shall see later that the action $S^{\mathrm{I}}$ (3.2) describes a GI theory. The Euler–Lagrange equations of motion obtained from $\mathscr{L}^{\mathrm{I}}$ (3.2) are

$$\left(\partial_0^2 - \partial_1^2\right)\phi = \big[e(\partial_0 + \partial_1)(A_1 - A_0) + (1-e)\left(\partial_0^2 - \partial_1^2\right)\theta$$
$$+ e\partial_1(\partial_1\theta - \partial_0\theta)\big] \tag{3.3a}$$

$$\partial_1(\partial_1 A_0 - \partial_0 A_1) = [e(\partial_0 + \partial_1)\phi + e^2(A_0 - A_1) + (e^2 - e)\partial_0\theta$$
$$- (e^2 + e)\partial_1\theta] \tag{3.3b}$$

$$\partial_0(\partial_1 A_0 - \partial_0 A_1) = [e(\partial_0 + \partial_1)\phi + e^2(A_0 - A_1) + 4e^2 A_1 + (e^2 - e)\partial_0\theta$$
$$+ (3e^2 - e)\partial_1\theta] \tag{3.3c}$$

$$\big[(1-e)\left(\partial_0^2 - \partial_1^2\right)\phi - (e-1)^2\partial_0^2\theta + (3e^2 - 2e + 1)\partial_1^2\theta\big]$$
$$= [(e^2 - e)\partial_0(A_0 - A_1) - (e^2 + e)\partial_1(A_0 - A_1)$$
$$- 4e^2\partial_1 A_1 - 2e^2\partial_0\partial_1\theta] \tag{3.3d}$$

With the help of these equations and the appropriate definition of the vector gauge current density $j^\mu$, we would be able to see in Section 4, that the four-divergence of the vector current density $j^\mu$ (i.e., $\partial_\mu j^\mu$) for this theory $S^{\mathrm{I}}$ (3.2) vanishes, and thereby implying that the theory $S^{\mathrm{I}}$ possesses (at the classical level) a local vector gauge symmetry (LVGS) or that it does not possess any vector gauge anomaly. Also, we will see in the next section that the theory $S^{\mathrm{I}}$ (3.2) describes a constrained theory possessing a set of three first-class constraints, implying in an alternative way that the system $S^{\mathrm{I}}$ (3.2) is GI. In the next section, we would also obtain explicitly the gauge transformations under which the system defined by the action $S^{\mathrm{I}}$ (3.2) remains invariant. In fact, we would be able to recover the physical content of the GNI theory defined by the action $S^{\mathrm{N}}$ (2.1), from the GI theory described by $S^{\mathrm{I}}$ (3.2), under some special choice of gauge (cf. Section 4). The quantum equivalence of the systems defined by the actions $S^{\mathrm{N}}$ (2.1) and $S^{\mathrm{I}}$ (3.2) (under the said special gauge), would be discussed in the next section.

## 4. THE HAMILTONIAN FORMULATION AND OPERATOR SOLUTION OF THE GI THEORY

### 4.1. The Hamiltonian Formulation

In this section, we consider the Hamiltonian formulation of the GI theory $\mathscr{L}^{\rm I}$ constructed from the GNI theory $\mathscr{L}^{\rm N}$ in the last section using the Stueckelberg method. The momenta canonically conjugate respectively to the field variables $A_0$, $A_1$, $\phi$ and $\theta$ obtained from the Lagrangian density $\mathscr{L}^{\rm I}$ (3.2) are

$$\Pi_0 := \frac{\partial \mathscr{L}^{\rm I}}{\partial(\partial_0 A_0)} = 0 \tag{4.1a}$$

$$E = \Pi^1 := \frac{\partial \mathscr{L}^{\rm I}}{\partial(\partial_0 A_1)} = (\partial_0 A_1 - \partial_1 A_0) \tag{4.1b}$$

$$\Pi := \frac{\partial \mathscr{L}^{\rm I}}{\partial(\partial_0 \phi)} = \partial_0\phi + e(A_0 - A_1) + (e-1)\partial_0\theta - e\partial_1\theta \tag{4.1c}$$

$$\begin{aligned}\Pi_\theta := \frac{\partial \mathscr{L}^{\rm I}}{\partial(\partial_0 \theta)} &= [(e-1)^2\partial_0\theta + (e-1)\partial_0\phi + e\partial_1\phi - e(A_0 - A_1) \\ &\quad + e^2(A_0 - A_1 - \partial_1\theta)] \qquad (4.1d)\\ &= [(e-1)\Pi + e\partial_1\phi - e\partial_1\theta] \qquad (4.1e)\end{aligned}$$

$\mathscr{L}^{\rm I}$ is thus seen to possess two primary constraints

$$\psi_1 = \Pi_0 \approx 0 \tag{4.2a}$$

$$\psi_2 = [\Pi_\theta - (e-1)\Pi - e\partial_1\phi + e\partial_1\theta] \approx 0 \tag{4.2b}$$

The canonical Hamiltonian density corresponding to the Lagrangian density $\mathscr{L}^{\rm I}$ (3.2) is

$$\begin{aligned}\mathscr{H}_{\rm c}^{\rm I} = \Bigg[&\frac{1}{2}(E^2 + \Pi^2) + \frac{1}{2}(\partial_1\phi)^2 + E\partial_1 A_0 - e(A_0 - A_1)(\Pi + \partial_1\phi - \partial_1\theta) \\ &+ 2e^2 A_1^2 + \frac{1}{2}(\partial_1\theta)^2 + (e-1)(\partial_1\phi)(\partial_1\theta) - e(2e+1)(\partial_1\theta)^2 \\ &+ e\Pi\partial_1\theta + 4e^2 A_1\partial_1\theta\Bigg]\end{aligned} \tag{4.3}$$

After implementing the primary constraints $\psi_1$ and $\psi_2$ in the canonical Hamiltonian density $\mathscr{H}_{\rm c}^{\rm I}$ with the help of Lagrange multiplier fields $u$ and $v$, the total Hamiltonian density $\mathscr{H}_{\rm T}^{\rm I}$ could be written as

$$\mathscr{H}_{\rm T}^{\rm I} = \mathscr{H}_{\rm c}^{\rm I} + \Pi_0 u + [(1-e)\Pi + \Pi_\theta - e\partial_1\phi + e\partial_1\theta]v \tag{4.4}$$

The Hamilton's equations of motion of the theory obtained from the total Hamiltonian $H_{\rm T}^{\rm I} = \int \mathscr{H}_{\rm T}^{\rm I}\, dx_1$ are

$$\partial_0\phi = \frac{\partial H_{\rm T}^{\rm I}}{\partial\Pi} = \Pi - e(A_0 - A_1) + e\partial_1\theta + (1-e)v \tag{4.5a}$$

$$-\partial_0\Pi = \frac{\partial H_{\rm T}^{\rm I}}{\partial\phi} = -\partial_1^2\phi + e\partial_1(A_0 - A_1) - (e-1)\partial_1^2\theta + e\partial_1 v \tag{4.5b}$$

$$\partial_0 A_0 = \frac{\partial H_{\rm T}^{\rm I}}{\partial\Pi_0} = u \tag{4.5c}$$

$$-\partial_0\Pi_0 = \frac{\partial H_{\rm T}^{\rm I}}{\partial A_0} = -\partial_1 E - e(\Pi + \partial_1\phi - \partial_1\theta) \tag{4.5d}$$

$$\partial_0 A_1 = \frac{\partial H_{\rm T}^{\rm I}}{\partial E} = E + \partial_1 A_0 \tag{4.5e}$$

$$-\partial_0 E = \frac{\partial H_{\rm T}^{\rm I}}{\partial A_1} = e(\Pi + \partial_1\phi - \partial_1\theta) + 4e^2(A_1 + \partial_1\theta) \tag{4.5f}$$

$$\partial_0\theta = \frac{\partial H_{\rm T}^{\rm I}}{\partial\Pi_\theta} = v \tag{4.5g}$$

$$-\partial_0\Pi_\theta = \frac{\partial H_{\rm T}^{\rm I}}{\partial\theta} = \big[-e\partial_1(A_0 - A_1) - \partial_1^2\theta - (e-1)\partial_1^2\phi + 2e(2e+1)\partial_1^2\theta - e\partial_1\Pi - 4e^2\partial_1 A_1 - e\partial_1 v\big] \tag{4.5h}$$

$$\partial_0 u = \frac{\partial H_{\rm T}^{\rm I}}{\partial\Pi_u} = 0 \tag{4.5i}$$

$$-\partial_0\Pi_u = \frac{\partial H_{\rm T}^{\rm I}}{\partial u} = \Pi_0 \tag{4.5j}$$

$$\partial_0 v = \frac{\partial H_{\rm T}^{\rm I}}{\partial\Pi_v} = 0 \tag{4.5k}$$

$$-\partial_0\Pi_v = \frac{\partial H_{\rm T}^{\rm I}}{\partial v} = [(1-e)\Pi + \Pi_\theta - e\partial_1\phi + e\partial_1\theta] \tag{4.5l}$$

For the PB $\{\, ,\}_{\rm P}$ of two functions $A$ and $B$ we choose the convention

$$\{A(x), B(y)\}_{\rm P} := \int dz \sum_\alpha \left[\frac{\partial A(x)}{\partial q_\alpha(z)}\frac{\partial B(y)}{\partial p_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)}\frac{\partial B(y)}{\partial q_\alpha(z)}\right] \tag{4.5}$$

Now the requirement of the preservation in time of the primary constraint $\psi_1$ leads to the secondary constraint:

$$\psi_3 := \left\{\psi_1, \mathscr{H}_{\rm T}^{\rm I}\right\}_{\rm P} = [\partial_1 E + e(\Pi + \partial_1\phi - \partial_1\theta)] \approx 0 \tag{4.6}$$

The preservation of $\psi_2$ and $\psi_3$ for all time, however, does not give rise to any further constraints. The theory is thus seen to possess three constraints $\psi_1$, $\psi_2$, and $\psi_3$. Also the matrix of the PBs of the constraints $\psi_i$ is seen to be singular, thereby implying that the constraints $\psi_i$ form a set of first-class constraints (Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a,b,c, 1994a,b,c, 1999; Kulshreshtha and Muller-Kirsten, 1992) and that the theory $\mathscr{L}^{\mathrm{I}}$ describes a gauge invariant theory. In fact the Lagrangian density $\mathscr{L}^{\mathrm{I}}$ is seen to be invariant under the time-dependent chiral gauge transformations:

$$\delta\phi = -\beta(x, t); \quad \delta A_0 = \partial_0\beta(x, t); \quad \delta A_1 = \partial_1\beta(x, t) \tag{4.7a}$$

$$\delta\theta = -\beta(x, t); \quad \delta u = \partial_0\partial_0\beta(x, t); \quad \delta v = -\partial_0\beta(x, t) \tag{4.7b}$$

$$\delta\Pi = \delta\Pi_0 = \delta E = \delta\Pi_\theta = \delta\Pi_u = \delta\Pi_v = 0 \tag{4.7c}$$

where $\beta(x, t)$ is an arbitrary function of the coordinates.

In quantizing the theory using Dirac's procedure (Dirac, 1950), one has to convert the set of first-class constraints of the theory into a set of second-class ones. One achieves this by imposing, arbitrarily, some additional constraints on the system in the form of gauge-fixing conditions. Following the work of Mitra and Rajaraman (1988), Girotti *et al.* (1986), Boyanovski (1987), and Falck and Kramer (1987) we go to a special gauge given by $\partial^\mu\theta = 0$ (or equivalently, $\partial_0\theta = \mathring{\theta} = 0$ and $-\partial_1\theta = -\theta' = 0$), and accordingly we choose the gauge-fixing conditions of the theory as (Boyanovski, 1987; Falck and Kramer, 1987; Girotti *et al.*, 1986; Mitra and Rajaraman 1988):

$$\mathscr{G}_1 = -\partial_1\theta \approx 0 \tag{4.8a}$$

$$\mathscr{G}_2 = [(A_0 + A_1) - (\Pi + \Pi_\theta) + e(\Pi + \partial_1\phi)] \approx 0 \tag{4.8b}$$

With the gauge-fixing conditions (4.8) the total set of constraints of the theory becomes

$$\xi_1 = \psi_1 = \Pi_0 \approx 0 \tag{4.9a}$$

$$\xi_2 = \psi_2 = [\Pi + \Pi_\theta - e(\Pi + \partial_1\phi) + e\partial_1\theta] \approx 0 \tag{4.9b}$$

$$\xi_3 = \psi_3 = [\partial_1 E + e(\Pi + \partial_1\phi - \partial_1\theta)] \approx 0 \tag{4.9c}$$

$$\xi_4 = \mathscr{G}_1 = -\partial_1\theta \approx 0 \tag{4.9d}$$

$$\xi_5 = \mathscr{G}_2 = [A_0 + A_1 - (\Pi + \Pi_\theta) + e(\Pi + \partial_1\phi)] \approx 0 \tag{4.9e}$$

The matrix of PBs of the constraints $\xi_i$ namely, $\boldsymbol{N}_{\alpha\beta}(z, z')\!: = \{\xi_\alpha(z), \xi_\beta(z')\}_{\mathrm{p}}$, is then calculated. The nonvanishing matrix elements of the matrix $\boldsymbol{N}_{\alpha\beta}(z, z')$ are

$$N_{15} = -N_{51} = -\delta(z - z') \tag{4.10a}$$

$$N_{22} = N_{33} = 2e^2\partial_1\delta(z - z') \tag{4.10b}$$

$$N_{23} = N_{32} = -2e^2\partial_1\delta(z - z') \tag{4.10c}$$

$$N_{25} = N_{52} = (2e^2 - 1)\partial_1\delta(z - z') \tag{4.10d}$$

$$N_{34} = N_{43} = -\partial_1\delta(z - z') \tag{4.10e}$$

$$N_{35} = N_{53} = -e(2e - 1)\partial_1\delta(z - z') \tag{4.10f}$$

$$N_{45} = N_{54} = +\partial_1\delta(z - z') \tag{4.10g}$$

$$N_{55} = 2e(e - 1)\partial_1\delta(z - z') \tag{4.10h}$$

The matrix $\boldsymbol{N}_{\alpha\beta}$ is seen to be nonsingular and therefore its inverse exists. The nonvanishing elements of the inverse of the matrix $\boldsymbol{N}_{\alpha\beta}$ (i.e., the elements of the matrix $(\boldsymbol{N}^{-1})\ \alpha\beta$ are

$$(N^{-1})_{11} = (1/2e^2)\partial_1\delta(z - z') \tag{4.11a}$$

$$(N^{-1})_{12} = -(N^{-1})_{21} = (1/2e^2)\delta(z - z') \tag{4.11b}$$

$$(N^{-1})_{13} = -(N^{-1})_{31} = \delta(z - z') \tag{4.11c}$$

$$(N^{-1})_{14} = -(N^{-1})_{41} = -(1 - e)\delta(z - z') \tag{4.11d}$$

$$(N^{-1})_{15} = -(N^{-1})_{51} = \delta(z - z') \tag{4.11e}$$

$$(N^{-1})_{22} = (1/4e^2)\ \epsilon\ (z - z') \tag{4.11f}$$

$$(N^{-1})_{24} = (N^{-1})_{42} = -\left(\frac{1}{2}\right)\epsilon\ (z - z') \tag{4.11g}$$

$$(N^{-1})_{34} = (N^{-1})_{43} = -\left(\frac{1}{2}\right)\epsilon\ (z - z') \tag{4.11h}$$

with

$$\int \boldsymbol{N}(x, z)\boldsymbol{N}^{-1}(z, y)dz = 1_{5\times5}\delta(x - y) \tag{4.12}$$

Here $\epsilon(z - z')$ is a step function defined as

$$\epsilon(z - z) := \begin{cases} +1, & (z - z') > 0 \\ -1, & (z - z') < 0 \end{cases} \tag{2.9}$$

Finally, the nonvanishing equal-time commutators of the GI theory described by $\mathscr{L}^{\mathrm{I}}$ under the gauge (4.8) are obtained as

$$[\phi(x), \Pi(y)] = \frac{3}{2}i\delta(x - y) \tag{4.13a}$$

$$[A_1(x), \Pi(y)] = \frac{-i}{2e}\delta(x - y) \tag{4.13b}$$

$$[A_1(x), E(y)] = i\delta(x - y) \tag{4.13c}$$

$$[A_0(x), \Pi(y)] = \frac{-i}{2e}\partial_1\delta(x-y) \tag{4.13d}$$

$$[A_0(x), E(y)] = -i\delta(x-y) \tag{4.13e}$$

$$[\phi(x), \phi(y)] = \frac{i}{4}\epsilon(x-y) \tag{4.13f}$$

$$[\phi(x), A_1(y)] = \frac{-i}{2e}\delta(x-y) \tag{4.13g}$$

$$[\phi(x), A_0(y)] = \frac{-i}{2e}\delta(x-y) \tag{4.13h}$$

$$[A_1(x), A_1(y)] = \left(\frac{i}{2e^2}\right)\partial_1\delta(x-y) \tag{4.13i}$$

$$[A_1(x), A_0(y)] = \left(\frac{i}{2e^2}\right)\partial_1\delta(x-y) \tag{4.13j}$$

$$[A_0(x), A_0(y)] = \left(\frac{i}{2e^2}\right)\partial_1\delta(x-y) \tag{4.13k}$$

$$[\Pi(x), \Pi(y)] = \frac{i}{2}\partial_1\delta(x-y) \tag{4.13l}$$

$$[\phi(x), \Pi_\theta(y)] = \left(\frac{i}{2}\right)\delta(x-y) \tag{4.13m}$$

$$[A_1(x), \Pi_\theta(y)] = \left(\frac{1-2e}{2e}\right)i\,\partial_1\delta(x-y) \tag{4.13n}$$

$$[A_0(x), \Pi_\theta(y)] = \left(\frac{1-2e}{2e}\right)i\,\partial_1\delta(x-y) \tag{4.13o}$$

$$[\theta(x), \Pi_\theta(y)] = 2i\delta(x-y) \tag{4.13p}$$

$$[A_0(x), \theta(y)] = 2i\delta(x-y) \tag{4.13q}$$

$$[\Pi(x), \Pi_\theta(y)] = -\frac{i}{2}\partial_1\delta(x-y) \tag{4.13r}$$

$$[\Pi_\theta(x), \Pi_\theta(y)] = \frac{i}{2}\partial_1\delta(x-y) \tag{4.13s}$$

Following the squence of reasoning offered in (Falck and Kramer, 1987; Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a, 1994a,b,c), it is easy to see that (4.13) together with $\mathscr{H}_c^{\rm I}$ (4.3) under the gauge (4.8) reproduce precisely the quantum system described by $\mathscr{L}^{\rm N}$ (2.1) (Falck and Kramer, 1987). The gauge (4.8) translates the GI version of the theory described by $\mathscr{L}^{\rm I}$ into the GNI one described by $\mathscr{L}^{\rm N}$. A comparision of (2.8) and (4.13) reveals that (4.13a)–(4.13l) coincide completely with (2.8) as they should. The additional commutators appearing in (4.13) (viz., (4.13m)–(4.13s)) express merely the dependence on $\theta$ and $\Pi_\theta$. Infact, the physical Hilbert spaces of the two theories ($\mathscr{L}^{\rm I}$ and $\mathscr{L}^{\rm N}$) are the same. The addition of the Stueckelberg term ($\mathscr{L}^{\rm S}$) to the theory (i.e., to $\mathscr{L}^{\rm N}$) enlarges only the unphysical part of the full Hilbert space of the theory $\mathscr{L}^{\rm N}$, without modifying the physical content of the theory. The Stueckelberg field $\theta$ itself, in fact, represents only an unphysical degree of freedom and correspondingly the physics of the theories with and without the Stueckelberg term remains the same (Falck and Kramer, 1987).

For the latter use (in the next section), for considering the BRST formulation of the gauge-invariant theory described by $\mathscr{L}^{\rm I}$, we convert the total Hamiltonian density $\mathscr{H}^{\rm I}_{\rm T}$ into the first-order Langrangian density

$$\mathscr{L}^{\rm I}_{\rm I0} = \Pi(\partial_0\phi) + E(\partial_0 A_1) + \Pi_0(\partial_0 A_0) + \Pi_\theta(\partial_0\theta) + \Pi_u(\partial_0 u) + \Pi_v(\partial_0 v) - \mathscr{H}^{\rm I}_{\rm T} \tag{4.14a}$$

$$= \Big[\Pi(\partial_0\phi) + E(\partial_0 A_1) + \Pi_u(\partial_0 u) + \Pi_v(\partial_0 v) - \frac{1}{2}(E^2 + \Pi^2 + (\partial_1\phi)^2) - E(\partial_1 A_0) + e(A_0 - A_1)(\Pi + \partial_1\phi) - 2e^2 A_1^2 - \frac{1}{2}(\partial_1\theta)^2 - (e-1)(\partial_1\phi)(\partial_1\theta) + e(1+2e)(\partial_1\theta)^2 - e(A_0 - A_1)\partial_1\theta - e\Pi\partial_1\theta - 4e^2 A_1\partial_1\theta - (\partial_0\theta)(\Pi - e\Pi - e\partial_1\phi + e\partial_1\theta)\Big] \tag{4.14b}$$

The generator of the LVGT is the charge operator of the theory:

$$J^0 = \int j^0\, dx \tag{4.15a}$$

$$j^0 = \big[(\partial_1\beta)(\partial_0 A_1 - \partial_1 A_0) - \beta[e\partial_0\phi + (e^2 - e)\partial_0\theta - (e^2 + e)\partial_1\theta + e\partial_1\phi + e^2(A_0 - A_1)]\big] \tag{4.15b}$$

The current operator of the theory is

$$J^1 = \int j^1\, dx \tag{4.16a}$$

$$j^1 = \big[\beta[(3e^2 - e)\partial_1\theta + (e^2 - e)\partial_0\theta + e\partial_0\phi + e\partial_1\phi + e^2 A_0 + 3e^2 A_1] - (\partial_0\beta)(\partial_0 A_1 - \partial_1 A_0)\big] \tag{4.16b}$$

The divergence of the vector current density, namely, $\partial_\mu j^\mu$ is therefore seen to vanish, implying that the theory possesses at the classical level a local vector gauge symmetry (LVGS).

### 4.2. The Operator Solution

In this section, we obtain the operator solution (Floreanini and Jackiw, 1987; Harada, 1990a,b; Harada and Isutsui, 1988; Kim *et al.*, 1990, 1991, 1992) of the GI theory (3.2) under the gauge $\mathcal{G}_i$ (4.8).

From the constraints (4.2) and (4.6) of the GI theory (3.2), we choose $\phi$, $\Pi$, $A_1$, and $E$ as the independent variables (IVs) of the theory and the remaining (dependent) variables (DVs) are expressed by these IVs as

$$\Pi_0 = 0 \tag{4.17a}$$

$$A_0 = -A_1 \tag{4.17b}$$

$$\Pi_\theta = -(\Pi + E') \tag{4.17c}$$

The nonvanishing equal-time (canonical as well as noncanonical) commutation relations of the theory under the gauge (4.8) are given by (4.13). The reduced Hamiltonian density $\mathcal{H}_R^I$ of the theory (3.2) (obtained by implementing the constraints of the theory strongly) expressed in terms of IVs could then be written as

$$\mathcal{H}_R^I = \left[\frac{1}{2}(E^2 + \Pi^2 + (\partial_1\phi)^2 - E(\partial_1 A_1) + 2eA_1(\Pi + \partial_1\phi) + 2e^2A_1^2\right] \tag{4.18}$$

The field equation derived from the Heisenberg equations are

$$\partial_0\phi = -i\left[\phi, H_R^I\right] = (\Pi + 2eA_1) \tag{4.19a}$$

$$\partial_0\Pi = -i\left[\Pi, H_R^I\right] = (\partial_1\partial_1\phi + 2e\partial_1 A_1) \tag{4.19b}$$

$$\partial_0 A_1 = -i\left[A_1, H_R^I\right] = (E - \partial_1 A_1) \tag{4.19c}$$

$$\partial_0 E = -i\left[E, H_R^I\right] = [-\partial_1 E - 2e(\Pi + \partial_1\phi) + 4e^2 A_1] \tag{4.19d}$$

where

$$H_R^I = \int dx\, \mathcal{H}_R^I \tag{4.20}$$

is the reduced Hamiltonian of the theory. Now using (4.17)–(4.20), we obtain the following equations:

$$\Box\phi = [-e(\partial_\mu A^\mu - \epsilon^{\mu\nu}\partial_\mu A_\nu)] \tag{4.21a}$$

$$\partial_\mu F^{\mu\nu} = [-2e^2 A^\nu - e(g^{\nu\alpha} + \epsilon^{\nu\alpha})\partial_\alpha\phi] \tag{4.21b}$$

$$[\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu] = 0 \tag{4.21c}$$

The equations (Eq. (4.21)) are now solved and the most general solution of (4.21) gives

$$\phi = [\sigma - h] \tag{4.22a}$$

$$A_\mu = \left(\frac{-1}{2e}\right)[\partial_\mu\phi + \epsilon_{\mu\nu}\partial^\nu\phi + 2\epsilon_{\mu\nu}\partial^\nu h] \tag{4.22b}$$

$$(\Box + m^2)\sigma = 0; \quad m = 2|e| \tag{4.22c}$$

$$\Box h = 0 \tag{4.22d}$$

where the free fields $\sigma$ and $h$ are expressed in terms of the IV's as

$$\sigma = \left[-\frac{1}{2e}E\right] \tag{4.23a}$$

$$h = \left[-\frac{1}{2e}E - \phi\right] \tag{4.23b}$$

Here the field $\sigma$ is a massive field and the field $h$ is a massless field as is evident from (4.22). The two-dimensional (unequal-time: $x_0 \neq y_0$) commutation relations involving the free fields $\sigma$ and $h$ are (with $x \equiv x^\mu$, $y = y^\mu$)

$$[\sigma(x), \sigma(y)] = i\Delta(x - y; m^2) \tag{4.24a}$$

$$[h(x), h(y)] = iD(x - y) = i\Delta(x - y; 0) \tag{4.24b}$$

where

$$\Delta(x - y; m^2) = (2\pi i)^{-1}\int d^2k\, \epsilon(k_0)\delta(k^2 - m^2)\, e^{-ikx} \tag{4.25a}$$

$$D(x - y) = \Delta(x - y; 0) \tag{4.25b}$$

and the propagator for $A_\mu$ is given by

$$iD_{\mu\nu}(k) = \int d^2x\, e^{ikx}\langle 0|T^* A_\mu(x)A_\nu(0)|0\rangle \tag{4.26}$$

where $A_\mu(x)$ is defined by (4.22).

## 5. THE BRST FORMULATION OF THE GI THEORY

### 5.1. BRST Invariance

For the BRST formulation of the theory, we rewrite the GI theory described by $\mathscr{L}^{\mathrm{I}}$ as a quantum system that possesses the generalized gauge invariance called BRST symmetry. For this, we enlarge the Hilbert space of our GI theory and introduce new anticommuting variables $c$ and $\bar{c}$ (which are grassmann numbers on the classical level and operators in the quantized theory) and a commuting variable $b$ such that (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998, 2000; Kulshreshtha *et al.*, 1993a,b,c, 1994a,b,c, 1999; Kulshreshtha and Kulshreshtha, 2001, in press; Kulshreshtha and Muller-Kirsten, 1992; Nameshchansky *et al.*, 1988; Tyutin, 1975)

$$\hat{\delta}\phi = -c;\ \hat{\delta}A_0 = \partial_0 c;\ \hat{\delta}A_1 = \partial_1 c;\ \hat{\delta}\theta = -c;\ \hat{\delta}u = \partial_0\partial_0 c \tag{5.1a}$$

$$\hat{\delta}v = -\partial_0 c \quad \hat{\delta}\Pi = \hat{\delta}E = \hat{\delta}\Pi_0 = \hat{\delta}\Pi_\theta = \hat{\delta}\Pi_u = \hat{\delta}\Pi_v = 0 \tag{5.1b}$$

$$\hat{\delta}c = 0; \qquad \hat{\delta}\bar{c} = b; \qquad \hat{\delta}b = 0 \tag{5.1c}$$

with the property $\hat{\delta}^2 = 0$. We then define a BRST-invariant function of the dynamical variables to be a function $f(\Pi, \Pi_0, E, \Pi_\theta, \Pi_b, \Pi_c, \Pi_{\bar{c}}, \phi, A_0, A_1, \theta, b, c, \bar{c})$ such that $\hat{\delta} f = 0$.

## 5.2. Gauge Fixing in the BRST Formalism

Performing gauge fixing in the BRST formalism implies adding to the first-order Lagrangian density (4.14) a function that is trivially BRST-invariant (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Nameshchansky *et al.*, 1988; Tyutin, 1975). We thus write the gauge-fixed quantum Lagrangian density (taking e.g., a trivial BRST-invariant function as follows) (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Nameshchansky *et al.*, 1988; Tyutin, 1975):

$$\begin{aligned}\mathscr{L}_{\mathrm{BRST}} = \Bigg[&\Pi(\partial_0\phi) + E(\partial_0 A_1) + \Pi_u(\partial_0 u) + \Pi_v(\partial_0 v) - \frac{1}{2}(E^2 + \Pi^2 + (\partial_1\phi)^2)\\ &- E(\partial_1 A_0) + e(A_0 - A_1)(\Pi + \partial_1\phi) - 2e^2 A_1^2 - \frac{1}{2}(\partial_1\theta)^2\\ &- (e-1)(\partial_1\phi)(\partial_1\theta) + e(1+2e)(\partial_1\theta)^2 - e(A_0 - A_1)\partial_1\theta\\ &- e\Pi\partial_1\theta - 4e^2 A_1\partial_1\theta - (\partial_0\theta)(\Pi - e\Pi - e\partial_1\phi + e\partial_1\theta)\\ &+ \hat{\delta}\left[\bar{c}\left(\partial_0 A_0 - A_1 - \phi - \partial_1\theta + \frac{1}{2}b\right)\right]\Bigg]\end{aligned} \tag{5.2}$$

The last term in the above equation is the extra BRST-invariant gauge-fixing term. Using the definition of $\hat{\delta}$ we can rewrite $\mathscr{L}_{\mathrm{BRST}}$ (with one integration by parts) as

$$\begin{aligned}\mathscr{L}_{\mathrm{BRST}} = \Bigg[&\Pi(\partial_0\phi) + E(\partial_0 A_1) + \Pi_u(\partial_0 u) + \Pi_v(\partial_0 v) - \frac{1}{2}(E^2 + \Pi^2 + (\partial_1\phi)^2)\\ &- E(\partial_1 A_0) + e(A_0 - A_1)(\Pi + \partial_1\phi) - 2e^2 A_1^2 - \frac{1}{2}(\partial_1\theta)^2\\ &- (e-1)(\partial_1\phi)(\partial_1\theta) + e(1+2e)(\partial_1\theta)^2 - e(A_0 - A_1)\partial_1\theta\end{aligned}$$

$$- e\Pi\partial_1\theta - 4e^2 A_1\partial_1\theta - (\partial_0\theta)(\Pi - e\Pi - e\partial_1\phi + e\partial_1\theta)$$
$$+ \frac{1}{2}b^2 - b(\phi + \partial_1\theta - \partial_0 A_0 + A_1) - \bar{c}c + (\partial_0\bar{c})(\partial_0 c)\Big] \tag{5.3}$$

Proceeding classically, the Euler–Lagrange equation for $b$ reads

$$b = \phi + (\partial_1\theta - \partial_0 A_0) + A_1 \tag{5.4}$$

Also, the requirement $\hat{\delta}b = 0$ (cf. (5.1)) implies

$$\hat{\delta}b = (\hat{\delta}\phi + \hat{\delta}\partial_1\theta - \hat{\delta}\partial_0 A_0 + \hat{\delta}A_1) = 0 \tag{5.5}$$

which in turn implies

$$-\partial_0\partial_0 c = c \tag{5.6}$$

The above equation is also an Euler–Lagrange equation obtained by the variation of $\mathscr{L}_{\text{BRST}}$ with respect to $\bar{c}$. In introducing the momenta we have to be careful in defining those for the fermionic variables. Thus we define the bosonic momenta in the usual way so that

$$\Pi_0 = \frac{\partial\,\mathscr{L}_{\text{BRST}}}{\partial(\partial_0 A_0)} = b \tag{5.7}$$

but the fermionic momenta are defined using the directional derivatives such that (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Nameshchansky *et al.*, 1988; Tyutin, 1975):

$$\Pi_c := \mathscr{L}_{\text{BRST}}\frac{\overleftarrow{\partial}}{\partial(\partial_0 c)} = \partial_0\bar{c}; \qquad \Pi_{\bar{c}} := \frac{\overrightarrow{\partial}}{\partial(\partial_0\bar{c})}\mathscr{L}_{\text{BRST}} = \partial_0 c \tag{5.8}$$

implying that the variable canonically conjugate to $c$ is $\partial_0\bar{c}$ and the variable conjugate to $\bar{c}$ is $\partial_0 c$. In constructing the quantum Hamiltonian density $\mathscr{H}_{\text{BRST}}$ from the Lagrangian density in the usual way, we remember that the former has to be Hermitian. Accordingly, we have (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Nameshchansky *et al.*, 1988; Tyutin, 1975)

$$\mathscr{H}_{\text{BRST}} = [\Pi(\partial_0\phi) + E(\partial_0 A_1) + \Pi_0(\partial_0 A_0) + \Pi_\theta(\partial_0\theta) + \Pi_u(\partial_0 u) + \Pi_v(\partial_0 v)$$
$$+ \Pi_c(\partial_0 c) + (\partial_0\bar{c})\Pi_{\bar{c}} - \mathscr{L}_{\text{BRST}}]$$
$$= \Big[\frac{1}{2}(E^2 + \Pi^2 + (\partial_1\phi)^2) + E(\partial_1 A_0) - e(\Pi + \partial_1\phi)(A_0 - A_1)$$

$$+ 2e^2 A_1^2 + \frac{1}{2}(\partial_1\theta)^2 + (e-1)(\partial_1\phi)(\partial_1\theta) - e(1+2e)(\partial_1\theta)^2$$

$$+ e(\partial_1\theta)(A_0 - A_1) + e\Pi\partial_1\theta + 4e^2 A_1(\partial_1\theta) + \Pi_0(\phi + \partial_1\theta + A_1)$$

$$\left. - \frac{1}{2}\Pi_0^2 + \Pi_c\Pi_{\bar{c}} + \bar{c}c \right] \tag{5.9}$$

We can check the consistency of (5.8) with (5.9) by looking at Hamilton's equations for the fermionic variables

$$\partial_0 c = \frac{\vec{\partial}}{\partial \Pi_c}\mathscr{H}_{\mathrm{BRST}}, \qquad \partial_0 \bar{c} = \mathscr{H}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial \Pi_{\bar{c}}} \tag{5.10}$$

thus we see that

$$\partial_0 c = \frac{\vec{\partial}}{\partial \Pi_c}\mathscr{H}_{\mathrm{BRST}} = \Pi_{\bar{c}}; \qquad \partial_0 \bar{c} = \mathscr{H}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial \Pi_{\bar{c}}} = \Pi_c \tag{5.11}$$

is in agreement with (5.8). For the operators $c$, $\bar{c}$, $\partial_0 c$, and $\partial_0\bar{c}$, one needs to specify the anticommutation relations of $\partial_0 c$ with $\bar{c}$ or of $\partial_0\bar{c}$ with $c$, but not of $c$ with $\bar{c}$. In general, $c$ and $\bar{c}$ are independent canonical variables and one assumes that (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Nameshchansky *et al.*, 1988; Tyutin, 1975)

$$\{\Pi_c, \Pi_{\bar{c}}\} = \{\bar{c}, c\} = 0 \tag{5.12a}$$

$$\frac{d}{dt}\{\bar{c}, c\} = 0 \quad \text{or} \quad \{\partial_0\bar{c}, c\} = -\{\partial_0 c, \bar{c}\} \tag{5.12b}$$

where $\{,\}$ means an anticommutator. We thus see that the anticommutators in (5.12b) are nontrivial and need to be fixed. To fix these, we demand that $c$ satisfy the Heisenberg equation (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Nameshchansky *et al.*, 1988; Tyutin, 1975):

$$[c, \mathscr{H}_{\mathrm{BRST}}] = i\partial_0 c \tag{5.13}$$

and using the property $c^2 = \bar{c}^2 = 0$ one obtains

$$[c, \mathscr{H}_{\mathrm{BRST}}] = \{\partial_0\bar{c}, c\}\partial_0 c \tag{5.14}$$

Equations (5.12)–(5.14) then imply:

$$\{\partial_0\bar{c}, c\} = -\{\partial_0 c, \bar{c}\} = i \tag{5.15}$$

Here the minus sign in the above equation implies the existence of states with negative norm in the space of state vectors of the theory (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 1998; Nameshchansky *et al.*, 1988; Tyutin, 1975).

The properties obeyed by fermionic variables could thus be summarized in a single equation (for further discussions) as

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\partial_0\bar{c}, \partial_0 c\} = 0; \quad \{\partial_0\bar{c}, c\} = i = -\{\partial_0 c, \bar{c}\} \tag{5.16}$$

## 5.3. The BRST Charge Operator

The BRST charge operator $Q$ is the generator of the BRST transformations (5.1). It is nilpotent and satisfies $Q^2 = 0$. It mixes operators that satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anticommutators with Fermi operators for the present theory satisfy

$$[\Pi, Q] = -e(\partial_1 c - \partial_0\partial_1 c) \tag{5.17a}$$

$$[\Pi_\theta, Q] = +e(\partial_1 c + \partial_0\partial_1 c) \tag{5.17b}$$

$$[\phi, Q] = -(e-1)\partial_0 c - ec; \qquad [\theta, Q] = +\partial_0 c \tag{5.17c}$$

$$[A_1, Q] = +\partial_1 c; \qquad [A_0, Q] = +\partial_0 c \tag{5.17d}$$

$$\{\bar{c}, Q\} = [e\Pi + e\partial_1\phi - e\partial_1\theta - \Pi_0 - \Pi - \Pi_\theta] \tag{5.17e}$$

$$\{\partial_0\bar{c}, Q\} = [e\partial_1\theta - \partial_1 E - e(\Pi + \partial_1\phi)] \tag{5.17f}$$

All other commutators and anticommutators involving $Q$ vanish. In view of (5.17), the BRST charge operator of the present gauge invariant theory can be written as

$$\begin{aligned} Q = \int dx\{&ic[\partial_1 E + e(\Pi + \partial_1\phi) - e\partial_1\theta] \\ &- i\partial_0 c[\Pi_0 + \Pi + \Pi_\theta - e(\Pi + \partial_1\phi) + e\partial_1\theta]\} \end{aligned} \tag{5.18}$$

This equation implies that the set of states satisfying the conditions (4.2) and (4.6) belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i.e., it belongs to the set of BRST-invariant states.

To understand the condition needed for recovering the physical states of the theory we rewrite the operators $c$ and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider (5.6). The solution of this equation (Eq. (5.6)) gives the Heisenberg operator $c(t)$ (and correspondingly $\bar{c}(t)$) as

$$c(t) = e^{it}B + e^{-it}D; \qquad \bar{c}(t) = e^{-it}B^\dagger + e^{it}D^\dagger \tag{5.19}$$

which at time $t = 0$ imply

$$c \equiv c(0) = B + D; \qquad \bar{c} \equiv \bar{c}(0) = B^\dagger + D^\dagger \tag{5.20a}$$

$$\mathring{c} \equiv \mathring{c}(0) = i(B - D); \qquad \mathring{\bar{c}} \equiv \mathring{\bar{c}}(0) = -i(B^\dagger - D^\dagger) \tag{5.20b}$$

By imposing the conditions (5.16), we now obtain the equations

$$B^2 + \{B, D\} + D^2 = B^{\dagger 2} + \{B^\dagger, D^\dagger\} + D^{\dagger 2} = 0 \tag{5.21a}$$

$$\{B, B^\dagger\} + \{D, D^\dagger\} + \{B, D^\dagger\} + \{B^\dagger, D\} = 0 \tag{5.21b}$$

$$\{B, B^\dagger\} + \{D, D^\dagger\} - \{B, D^\dagger\} - \{B^\dagger, D\} = 0 \tag{5.21c}$$

$$\{B, B^\dagger\} - \{D, D^\dagger\} - \{B, D^\dagger\} + \{D, B^\dagger\} = -1 \tag{5.21d}$$

$$\{B, B^\dagger\} - \{D, D^\dagger\} + \{B, D^\dagger\} - \{D, B^\dagger\} = -1 \tag{5.21e}$$

with the solution

$$B^2 = D^2 = B^{\dagger 2} = D^{\dagger 2} = 0 \tag{5.22a}$$

$$\{B, D\} = \{B^\dagger + D\} = \{B, D^\dagger\} = \{B^\dagger, D^\dagger\} = 0 \tag{5.22b}$$

$$\{B^\dagger, B\} = -\frac{1}{2}; \qquad \{D^\dagger, D\} = \frac{1}{2} \tag{5.22c}$$

We now let $|0\rangle$ denote the fermionic vacuum for which

$$B|0\rangle = D|0\rangle = 0 \tag{5.23}$$

Defining $|0\rangle$ to have norm one, (5.22c) implies

$$\langle 0|BB^\dagger|0\rangle = -\frac{1}{2}; \qquad \langle 0|DD^\dagger|0\rangle = +\frac{1}{2} \tag{5.24}$$

so that

$$B^\dagger|0\rangle \neq 0; \qquad D^\dagger|0\rangle \neq 0 \tag{5.25}$$

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathscr{H}_{\text{BRST}}$ is, however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators

$$\begin{aligned}\mathscr{H}_{\text{BRST}} = {} & \frac{1}{2}(E^2 + \Pi^2 + (\partial_1\phi)^2) + E\partial_1 A_0 - e(\Pi + \partial_1\phi)(A_0 - A_1) + 2e^2 A_1^2 \\ & + \frac{1}{2}(\partial_1\theta)^2 + (e-1)(\partial_1\phi)(\partial_1\theta) - e(1+2e)(\partial_1\theta)^2 \\ & + e(\partial_1\theta)(A_0 - A_1) + e\Pi\partial_1\theta + 4e^2 A_1\partial_1\theta + \Pi_0(\phi + \partial_1\theta + A_1) \\ & - \frac{1}{2}\Pi_0^2 + 2(B^\dagger B + D^\dagger D)\end{aligned} \tag{5.26}$$

and the BRST charge operator $Q$ is

$$\begin{aligned} Q = \int dx\{iB[(\partial_1 E + e(\Pi + \partial_1\phi) - e\partial_1\theta) - i(\Pi_0 + \Pi + \Pi_\theta \\ - e(\Pi + \partial_1\phi) + e\partial_1\theta)] + \; iD[(\partial_1 E + e(\Pi + \partial_1\phi) - e\partial_1\theta) \\ + i(\Pi_0 + \Pi + \Pi_\theta - e(\Pi + \partial_1\phi) + \; e\partial_1\theta)]\} \end{aligned} \tag{5.27}$$

Now because $Q|\psi\rangle = 0$, the set of states annihilated by $Q$ contains not only the set of states for which (4.2) and (4.6) hold but also additional states for which $B|\psi\rangle = D|\psi\rangle = 0$ and for which (4.2), and (4.6) do not hold. However, the Hamiltonian is also invariant under the anti-BRST transformation given by

$$\bar{\hat{\delta}}\phi = \bar{c};\; \bar{\hat{\delta}}A_0 = -\partial_0\bar{c};\; \bar{\hat{\delta}}A_1 = -\partial_1\bar{c};\; \bar{\hat{\delta}}\theta = \bar{c};\; \bar{\hat{\delta}}u = -\partial_0\partial_0 c \tag{5.28a}$$

$$\bar{\hat{\delta}}v = \partial_0 c, \quad \bar{\hat{\delta}}\Pi = \bar{\hat{\delta}}E = \bar{\hat{\delta}}\Pi_0 = \bar{\hat{\delta}}\Pi_\theta = \bar{\hat{\delta}}\Pi_u = \bar{\hat{\delta}}\Pi_v = 0 \tag{5.28b}$$

$$\bar{\hat{\delta}}\bar{c} = 0; \qquad \bar{\hat{\delta}}c = -b; \qquad \bar{\hat{\delta}}b = 0 \tag{5.28c}$$

with the generator or anti-BRST charge

$$\begin{aligned} \bar{Q} &= \int dx\{-i\bar{c}[(\partial_1 E + e(\Pi + \partial_1\phi) - e\partial_1\theta] + i\partial_0\bar{c}[\Pi_0 + \Pi + \Pi_\theta \\ &\quad - e(\Pi + \partial_1\phi) + e\partial_1\theta]\} \\ &= \int dx\{-iB^\dagger[(\partial_1 E + e(\Pi + \partial_1\phi) - e\partial_1\theta) - i(\Pi_0 + \Pi + \Pi_\theta - e(\Pi + \partial_1\phi) \\ &\quad + e\partial_1\theta)] - iD^\dagger[(\partial_1 E + e(\Pi + \partial_1\phi) - e\partial_1\theta) + i(\Pi_0 + \Pi + \Pi_\theta \\ &\quad - e(\Pi + \partial_1\phi) + e\partial_1\theta)]\} \end{aligned} \tag{5.29}$$

We also have

$$\partial_0 Q = [Q, H_{\text{BRST}}] = 0 \tag{5.30a}$$

$$\partial_0 \bar{Q} = [\bar{Q}, H_{\text{BRST}}] = 0 \tag{5.30b}$$

with

$$H_{\text{BRST}} = \int dx\; \mathscr{H}_{\text{BRST}} \tag{5.30c}$$

and we further impose the dual condition that both $Q$ and $\bar{Q}$ annihilate physical states, implying that

$$Q|\psi\rangle = 0 \quad \text{and} \quad \bar{Q}|\psi\rangle = 0 \tag{5.31}$$

The states for which (4.2) and (4.6) hold strongly satisfy both of these conditions and, in fact, are the only states satisfying both conditions, since although with (5.22)

$$2(B^\dagger B + D^\dagger D) = -2(BB^\dagger + DD^\dagger) \tag{5.32}$$

there are no states of this operator with $B^\dagger|0\rangle = 0$ and $D^\dagger|0\rangle = 0$ (cf. 5.25)), and hence no free eigenstates of the fermionic part of $H_{\rm BRST}$ which are annihilated by each of $B$, $B^\dagger$, $D$, $D^\dagger$. Thus the only states satisfying (5.31) are those satisfying the constraints (4.2) and (4.6). Further, the states for which (4.2) and (4.6) satisfy both of these conditions (5.31) and, in fact, are the only states satisfying both of these conditions (5.31), because in view of (5.20) one cannot have simultaneously $c$, $\partial_0 c$ and $\bar{c}$, $\partial_0\bar{c}$, applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (5.31) are those that satisfy the constraints of the theory (4.2) and (4.6) and they belong to the set of BRST-invariant and anti-BRST-invariant states.

Alternatively, in terms of fermionic annihilation and creation operators, the condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by $Q$ contains not only the states for which (4.2) and (4.6) holds but also additional states for which $\Pi_0|\psi\rangle \neq 0$, $[\Pi + \Pi_\theta - e(\Pi + \partial_1\phi) + e\partial_1\theta]|\psi\rangle \neq 0$, and $[\partial_1 E + e(\Pi + \partial_1\phi - \partial_1\theta)]|\psi\rangle \neq 0$. However $\underline{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\underline{Q}$ contains only the states for which $\Pi_0|\psi\rangle = 0$, $[\Pi + \Pi_\theta - e(\Pi + \partial_1\phi) + e\partial_1\theta]|\psi\rangle = 0$, and $[\partial_1 E + e(\Pi + \partial_1\phi - \partial_1\theta)]|\psi\rangle = 0$, simply because $B^\dagger|\psi\rangle \neq 0$ and $D^\dagger|\psi\rangle \neq 0$. Thus in this alternative way we also see that the states satisfying $Q|\psi\rangle = \underline{Q}|\psi\rangle = 0$ (i.e., satisfying (5.31)) are only those states that satisfy the constraints of the theory and also that these states belong to the set of BRST-invariant and anti-BRST-invariant states.

## 6. CONCLUSIONS AND DISCUSSIONS

In this work we have constructed a GI theory corresponding to a GNI-CSM with the Faddeevian regularization due to Mitra (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b) through the so-called Stueckelberg term (Kulshreshtha, 1998; Kulshreshtha *et al.*, 1993a, 1994a,b,c; Stueckelberg, 1941, 1957), the addition of which to the action of the original GNI theory restores the gauge-invariance to the theory. The canonical structure, constrained dynamics, and the Hamiltonian and BRST quantization and the operator solutions of this newly constructed GI theory obtained by the inclusion of the Stueckelberg term to the action of the GNI theory have been studied in the conventional form of dynamics on the hyperplanes $x^0 =$ constant, called the instant form of dynamics (Dirac, 1949). This theory has been studied in a recent paper (Kulshreshtha, 1998) on the hyperplanes of the light front $(x^0 + x^1) =$ constant, describing the front form of dynamics.

The original theory à la Mitra (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b) is seen to be GNI in both the forms of dynamics, namely, in the instant form, as seen in the present work, as well as in the light-front frame (or the FF) (Kulshreshtha, 1998). In both the cases, the theory possesses *a set of* second-class constraints (Kulshreshtha, 1998; Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b) and has a nonzero divergence of the vector gauge current implying that the theory does not have a local vector gauge symmetry (or that it is gauge-anomalous) in both the forms (IF and FF) of dynamics (Kulshreshtha, 1998; Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b). The FF theory has been studied in Kulshreshtha (1998), where we have constructed a GI theory from the corresponding GNI theory through the Stueckelberg term (Kulshreshtha, 1998). The Hamiltonian and BRST quantizations of this FF–GI theory have also been studied in Kulshreshtha (1998). The physical content of the original GNI theory in FF has also been recovered from that of the FF-GI theory under a special choice of gauge in Kulshreshtha (1998). In the present work, which is in the IF of dynamics, we see that the original GNI theory due to Mitra (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b) possesses a set of three second-class constraints that make the theory gauge anomalous (Mitra, 1992; Mukhopadhyay and Mitra, 1995a,b) (cf. Section 2). On the other hand, the GI theory as constructed in Section 3 through the Stueckelberg term is seen to possess a set of three first-class constraints. It is also seen to have a zero divergence for the vector gauge current, signifying that the theory is no longer gauge-anomalous and also that the vector gauge symmetry has been restored to the theory (cf. Section 3). The Hamiltonian quantization and the operator solutions of the newly constructed GI theory have been studied under a special choice of gauge given by (4.8). The total set of constraints (given by (4.9)) of this new GI theory under the gauge (4.8) clearly becomes second class and the Dirac quantization of this GI theory could therefore be achieved under the gauge (4.8). The results of the Hamiltonian quantization of the GI theory (3.2) under the gauge (4.8) are expressed in terms of the nonvanishing equal-time commutators (4.13). Further, it is easy to see that the GI theory $\mathscr{L}^{\rm I}$ (3.2) with (4.13) and $\mathscr{H}_{\rm c}^{\rm I}$ (4.3) under the gauge (4.8) reproduce precisely the quantum system described by $\mathscr{L}^{\rm N}$ (2.1) (Falck and Kramer, 1987). The gauge (4.8) thus translates the GI version of the theory described by $\mathscr{L}^{\rm I}$ into the GNI one described by $\mathscr{L}^{\rm N}$ (Falck and Kramer, 1987). Also, as observed in Section 3, the physical Hilbert spaces of the theories $\mathscr{L}^{\rm I}$ and $\mathscr{L}^{\rm N}$ are the same, because the Stueckelberg field $\theta$ represents only an unphysical degree of freedom (Falck and Kramer, 1987). Also, in the above Hamiltonian quantization of the theory under the gauge-fixing conditions one necessarily destroys the gauge invariance of the theory by fixing the gauge. In view of this, to achieve the quantization of the above GI theory such that the gauge invariance of the theory is maintained even under gauge-fixing, we go to a more generalized procedure called the BRST quantization (Becchi *et al.*, 1974; Henneaux, 1985; Kulshreshtha, 2000; Kulshreshtha *et al.*, 1993a,b,c, 1994a,b,c, 1999; Kulshreshtha and Kulshreshtha, 2001; Kulshreshtha and Mueller-Kirsten, 1992; Nameshchansky *et al.*, 1988; Tyutin, 1975). The BRST quantization of the present GI theory (3.2) constructed in Section 3 (Corresponding to the GNI theory (2.1)) under some specific gauge choice has finally been studied in Section 5. In this procedure, the BRST-quantized theory continues to possess the generalized gauge invariance called the BRST symmetry even under the BRST gauge fixing (cf. Section 5).

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